

Casimir stress on parallel plates in de Sitter space with signature change

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Abstract

The Casimir stress on two parallel plates in a de Sitter background corresponding to different metric signatures and cosmological constants is calculated for massless scalar fields satisfying Robin boundary conditions on the plates. Our calculation shows that for the parallel plates with false vacuum between and true vacuum outside, the total Casimir pressure leads to an attraction of the plates at very early universe.

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1 Introduction

The Casimir effect is one of the most interesting manifestations of non-trivial properties of the vacuum state in quantum field theory [1,2]. Since its first prediction by Casimir in 1948[2] this effect has been investigated for different fields having different boundary geometries [3, 8, 9, 10, 11, 12, 13, 14]. The Casimir effect can be viewed as the polarization of vacuum by boundary conditions or geometry. Therefore, vacuum polarization induced by a gravitational field is also considered as a Casimir effect.

Casimir effect may have interesting implications for the early universe. In [15] the Casimir effect of a massless scalar field with Dirichlet boundary condition in a spherical shell having different vacua inside and outside, which represents a bubble in the early universe with false/true vacuum inside/outside, has been investigated. The Casimir stress on two concentric spherical shells with constant comoving radius having different vacua inside and outside in de Sitter space, which corresponds to a spherically symmetric domain wall with thickness, has been calculated in [16].

In the context of hot big bang cosmology, the unified theories of the fundamental interactions predict that the universe passes through a sequence of phase transitions. Different types of topological objects may have been formed during these phase transitions, these include domain walls, cosmic strings and monopoles [17, 18, 19]. These topological defects appear as a consequence of breakdown of local or global gauge symmetries of a system composed by self-coupling iso-scalar Higgs fields Φ^a .

On the other hand, signature changing space-times have recently been of particular importance as the specific geometries with interesting physical effects. The original idea of signature change was due to Hartle, Hawking and Sakharov [20, 21]. This interesting idea would make it possible to have both Euclidean and Lorentzian metrics in path integral approach to quantum gravity. Latter it was shown that the signature change may happen even in classical general relativity, as well [22]-[32]. The issue of propagation of quantum fields on signature-changing space-times has also been of some interest [33]-[38]. For example, Dray *et al* have shown that the phenomenon of particle production may happen for scalar particles propagation in space-time with heterotic signature. They have also obtained a rule for propagation of massless scalar fields on a two-dimensional space-time with signature change. Dynamical determination of the metric signature in space-time of nontrivial topology is another interesting issue which has been studied in Ref.[39].

To the author knowledge, no attempt has been done to study the Casimir effect within the geometries with signature change. A relevant work to the present paper is Ref. [40]. In this work the Casimir effect for the free massless scalar field propagating on a two-dimensional cylinder with a signature-changing strip has been studied. Motivated by this new element in studying the Casimir effect we have paid attention to study such a non-trivial effect in a model of two parallel plates in de Sitter space with different cosmological constants and metric signatures between and outside.

2 Parallel Plates with Robin boundary conditions in de Sitter Space

Consider a massless scalar field coupled conformally to a de Sitter background space. The scalar field satisfies the following Robin boundary conditions on two parallel plates within

an arbitrary space-time [41]:

$$(1 + \beta_m(-1)^{m-1}\partial_x)\phi(x^\nu)|_{x=a_m} = 0, \quad m = 1, 2, \quad (1)$$

Here we have assumed that the two plates are normal to the Cartesian x -axis at $x = a_{1,2}$. The Robin boundary condition may be interpreted as the boundary condition on a thick plate [42]. Rewriting(1) in the following form:

$$\partial_x\phi(x^\nu) = (-1)^m \frac{1}{\beta_m}\phi(x^\nu), \quad (2)$$

where $|\beta_m|$, having the dimension of a length, may be called the skin-depth parameter. This is similar to the case of penetration of an electromagnetic field into a real metal, where the tangential component of the electric field is proportional to the skin-depth parameter.

The corresponding field equation has the form

$$(\nabla_\mu \nabla^\mu + \xi R)\phi = 0, \quad \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu), \quad (3)$$

where ∇_μ is the covariant derivative operator, and R is the Ricci scalar for de Sitter space. In the conformally coupled case, the corresponding stress-energy tensor is defined as [43]

$$T_{\mu\nu} = (1 - 2\xi)\partial_\mu\phi\partial_\nu\phi + (2\xi - 1/2)g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi - 2\xi\phi\nabla_\mu\nabla_\nu\phi + \frac{1}{12}g_{\mu\nu}\phi\nabla_\lambda\nabla^\lambda\phi, \quad (4)$$

where $\xi = \frac{1}{6}$. It is known that in the Minkowski space-time for the conformally coupled scalar field the perpendicular pressure, P , is uniform in the region between the plates and is given by [41]

$$P = 3\varepsilon_c, \quad (5)$$

where ε_c is the Casimir energy density. This Casimir energy has been calculated to be

$$\varepsilon = \varepsilon_c = \frac{-A}{8\Gamma(5/2)\pi^{3/2}a^4}, \quad (6)$$

where A depends on $\beta_{1,2}$ and a may be inferred from Eq.(4.15) of [41]. It has also been shown that for $\beta_1 = -\beta_2$

$$\varepsilon = \varepsilon_c = \frac{-\zeta_R(4)\Gamma(2)}{(4\pi)^2a^4} = \frac{-\pi^2}{1440a^4}, \quad (7)$$

which is the same as for the Dirichlet and Neumann boundary conditions.

Consider now two parallel plates in a de Sitter space-time. To make the maximum use of the flat-space calculations we use the de Sitter metric in the conformally flat form:

$$ds^2 = \frac{\alpha^2}{\eta^2}[d\eta^2 - \sum_{i=1}^3(dx^i)^2], \quad (8)$$

where η is the conformal time:

$$-\infty < \eta < 0. \quad (9)$$

The relation between the parameter α and the cosmological constant Λ is given by

$$\alpha^2 = \frac{3}{\Lambda}. \quad (10)$$

The quantization of a scalar field on background of the metric (8) is straightforward. Let $\{\phi_k(x), \phi_k^*(x)\}$ be a complete set of orthonormalized positive and negative frequency solutions to the field equation (3), obeying boundary condition (1). By expanding the field operator over these eigenfunctions, using the standard commutation rules and the definition of the vacuum state for the vacuum expectation values of the energy-momentum tensor one obtains [43]

$$\langle 0|T_\mu^\nu|0\rangle = \sum_k T_\mu^\nu\{\phi_k, \phi_k^*\}, \quad (11)$$

where $|0\rangle$ is the amplitude for the corresponding vacuum state, and the bilinear form $T_\mu^\nu\{\phi, \phi^*\}$ on the right is determined by the classical energy-momentum tensor (4). In the problem under consideration we have a conformally trivial situation, namely a conformally invariant field on background of the conformally flat spacetime. Instead of evaluating Eq.(11) directly on background of the curved metric, the vacuum expectation values can be obtained from the corresponding flat spacetime results for a scalar field ϕ by using the conformal properties of the problem under consideration. Under the conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ the ϕ field will be changed by the rule

$$\tilde{\phi}(x) = \Omega^{-1}\phi(x). \quad (12)$$

Using the standard relation between the energy-momentum tensor for conformally coupled situations [43]

$$\langle T_\nu^\mu[\tilde{g}_{\alpha\beta}] \rangle = \left(\frac{g}{\tilde{g}}\right)^{\frac{1}{2}} \langle T_\nu^\mu[g_{\alpha\beta}^{(M)}] \rangle - \frac{1}{2880\pi^2} \left[\frac{1}{6} \tilde{H}_\nu^{(1)\mu} - \tilde{H}_\nu^{(3)\mu} \right], \quad (13)$$

where $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are conformal to each other, with their respective determinants g and \tilde{g} . We are going to assume that $g_{\mu\nu}$ is the Minkowski metric. Now, $\langle T_\nu^\mu[g_{\alpha\beta}^{(M)}] \rangle$, the regularized energy-momentum tensor for a conformally coupled scalar field in the case of a parallel plate configuration in flat space-time is given by

$$\langle T_\nu^\mu[g_{\alpha\beta}^{(M)}] \rangle = \text{diag}(\varepsilon, -P, -P_\perp, -P_\perp) = \text{diag}(\varepsilon, 3\varepsilon, -\varepsilon, -\varepsilon). \quad (14)$$

The second term in (13) is the vacuum polarization due to the gravitational field, without any boundary conditions. The functions $H_\nu^{(1,3)\mu}$ are some combinations of curvature tensor components (see [43]). For a massless scalar field in de Sitter space, the term is given by [43, 46]

$$-\frac{1}{2880} \left[\frac{1}{6} \tilde{H}_\nu^{(1)\mu} - \tilde{H}_\nu^{(3)\mu} \right] = \frac{1}{960\pi^2\alpha^4} \delta_\nu^\mu. \quad (15)$$

From (5,7,10,13,14) one can obtain the vacuum pressure due to the boundary acting on the plates:

$$P_B^{(1,2)} = P_B(x_{1,2}) = \left(\frac{g}{\tilde{g}}\right)^{\frac{1}{2}} \left(\frac{-3\pi^2}{1440a^4}\right) = \frac{\eta^4}{\alpha^4} \left(\frac{-3\pi^2}{1440a^4}\right) = \frac{-\eta^4\Lambda^2}{3} \frac{\pi^2}{1440a^4}, \quad (16)$$

which is attractive. It has been shown that this pressure is zero for $x < a_1$ and $x > a_2$ [44, 45]. The gravitational part of the pressure according to (15) is equal to

$$P_G = - \langle T_1^1 \rangle = \frac{-1}{960\pi^2\alpha^4}. \quad (17)$$

This is the same from both sides of the plates, and hence leads to zero effective force. Therefore, the effective force acting on the plates is only given by the boundary part.

3 Parallel plates with different cosmological constants and signatures between and outside

Now, assume that there are different vacua between and outside of the plates, corresponding to α_{betw} and α_{out} in the metric(8). As we have seen in the previous section, the vacuum pressure due to the boundary is only non-vanishing between the plates. Therefore, we have for the pressure due to the boundary

$$P_B^{(1,2)} = \frac{\eta^4}{\alpha_{betw}^4} \frac{-3\pi^2}{1440a^4} = \frac{-\eta^4\Lambda_{betw}^2}{3} \frac{\pi^2}{1440a^4}. \quad (18)$$

Now, we obtain the pure effect of vacuum polarization due to the gravitational field without any boundary conditions in Euclidean (outside) region with the following metric

$$ds^2 = -\frac{\alpha^2}{\eta^2} [d\eta^2 + \sum_{i=1}^3 (dx^i)^2]. \quad (19)$$

To this end, we calculate the renormalized stress tensor for the massless scalar field in de Sitter space with Euclidean signature. We use Eq.(13), then we obtain $\langle T_\nu^\mu [g_{\alpha\beta}^{(M)}] \rangle = 0$, and

$$\begin{aligned} {}^{(1)}H_\nu^\mu &= 0, \\ {}^{(3)}H_\nu^\mu &= \frac{3}{\alpha^4} \delta_\nu^\mu. \end{aligned}$$

Therefore

$$\langle T_\nu^\mu [\tilde{g}_{\alpha\beta}] \rangle = \frac{1}{960\pi^2\alpha^4} \delta_\nu^\mu, \quad (20)$$

which is exactly the same result for the Lorentzian case [43]. The corresponding effective pressures for the Euclidean (outside) and Lorentzian (between) regions with α_{out} and α_{betw} , due to pure effect of gravitational vacuum polarization without any boundary condition, are given respectively by

$$P_{out}^E = - \langle T_1^1 \rangle_{out} = \frac{-1}{960\pi^2\alpha_{out}^4} = \frac{-\Lambda_{out}^2}{9} \frac{1}{960\pi^2}. \quad (21)$$

$$P_{betw}^L = - \langle T_1^1 \rangle_{betw} = \frac{-1}{960\pi^2\alpha_{betw}^4} = \frac{-\Lambda_{betw}^2}{9} \frac{1}{960\pi^2}, \quad (22)$$

The corresponding gravitational pressure on the plates is then given by

$$P_G = P_{betw}^L - P_{out}^E = \frac{-1}{9 \times 960\pi^2} (\Lambda_{betw}^2 - \Lambda_{out}^2). \quad (23)$$

We now proceed to calculate the stress due to the boundary effects P_B . The stress on the plates due to boundary effects for the Lorentzian metric (8) has been obtained as (16). In signature changing case we have correspondingly

$$P_B^{L-E} = \langle 0|T_x^x|0\rangle_{betw}^L - \langle 0|T_x^x|0\rangle_{out}^E. \quad (24)$$

The scalar field $\phi(\vec{x}, \eta)$ in the Lorentzian de Sitter space satisfies

$$(\square + \xi R)\phi(\vec{x}, \eta) = 0, \quad (25)$$

where \square is the Laplace-Beltrami operator for de Sitter metric, and ξ is the coupling constant. For conformally coupled field in four dimension $\xi = \frac{1}{6}$, and R , the Ricci scalar curvature, is given by

$$R = 12\alpha^{-2}. \quad (26)$$

Taking into account the separation of variables as

$$\phi(\vec{x}, \eta) = A(\vec{x})T(\eta), \quad (27)$$

for the inside Lorentzian domain with

$$T_L(\eta) = \frac{1}{\sqrt{2\omega(2\pi)^3}} \exp^{-i\omega\eta}, \quad (28)$$

the corresponding Euclidean η -dependence takes on the form

$$T_E(\eta) = \frac{1}{\sqrt{2\omega(2\pi)^3}} \exp^{-\omega\eta}, \quad (29)$$

for the scalar field to be normalizable in η . Based on the following mode expansion

$$\phi = \sum_i (a_i^- \varphi_i^- + a_i^+ \varphi_i^+), \quad (30)$$

and normal ordering $\langle 0|a_i^- a_i^+|0\rangle = 1$, the detailed calculations, using Eqs.(11),(28) and (29) in Eq.(24), lead to

$$P_B^{L-E} = \frac{-\eta^4 \Lambda_{betw}^2}{3} \frac{\pi^2}{1440a^4} + \frac{\eta^2}{576\pi^5} (\Lambda_{betw}^2 - \Lambda_{out}^2) \zeta(2). \quad (31)$$

where $\zeta(2)$ is the Zeta function and Abel-Plana summation formula has been used to regularize the infinite sum $\sum_i \omega_i$. Taking into account the gravitational pressure on the plates we obtain the total result

$$P = P_G + P_B = \frac{-1}{9 \times 960\pi^2} (\Lambda_{betw}^2 - \Lambda_{out}^2) - \frac{\eta^4 \Lambda_{betw}^2}{3} \frac{\pi^2}{1440a^4} + \frac{\eta^2}{576\pi^5} (\Lambda_{betw}^2 - \Lambda_{out}^2) \zeta(2) \quad (32)$$

The first term P_G which is pure gravitational effect without boundary is η -independent, but is sensitive to the initial (in terms of η) values of Λ_{betw} and Λ_{out} . Given a false vacuum between the plates, and true vacuum out-side, i.e. $\Lambda_{betw} > \Lambda_{out}$, then the gravitational part is negative, the second term is always negative, but in this case the last term is positive. At very early universe, namely $\eta \simeq 0$ the constant first term dominates and there is an attraction between the plates. On the other hand, at times $\eta > 1$, the first term is ignorable and there is a competition between the second (negative) and third (positive) terms. For the case of true vacuum between the plates and false vacuum out-side, i.e. $\Lambda_{betw} < \Lambda_{out}$, the gravitational pressure is positive, and another terms are negative. Therefore, the total pressure may be either negative or positive. In this case, at very early universe, i.e. $\eta \simeq 0$, the total pressure $P > 0$, this initial repulsion of the parallel plates will stopped at times $\eta > 1$.

4 Conclusion

In the present paper we have investigated the Casimir effect for a conformally coupled massless scalar field confined in the region between two parallel plates in a de Sitter background corresponding to different metric signatures and cosmological constants between and outside of the plates. The general case of the mixed(Robin) boundary conditions is considered. The vacuum expectation values of the energy-momentum tensor are derived from the corresponding flat spacetime results by using the conformal properties of the problem. Previously this method has been used in [14] to investigate the vacuum characteristics of the Casimir configuration on a background of conformally flat brane-world geometries for a massless scalar field with Robin boundary conditions on plates. Also this method has been used in [44] to derive the vacuum characteristics of the Casimir configuration on a background of static domain wall geometry for a scalar field with Dirichlet boundary condition on plates. (For investigations of the Casimir energy in brane-world models with de Sitter branes, see Refs. [48]-[53]).

Our calculation shows that for the parallel plates with false vacuum between and true vacuum outside, the total Casimir pressure leads to an attraction of the plates at very early universe, namely $\eta \simeq 0$. The boundary term is proportional to the fourth power of the inverse distance between the plates, and is always negative, which means a huge attractive force for small distances. In contrast, plates with a true vacuum between them may repel each other to a maximum distance and attract again. The result may be of interest in the case of formation of cosmic domain walls in the early universe, where the wall orthogonal to the x -axis is described by the function $\phi_i(x)$ interpolating between two different minima at $x \rightarrow \pm\infty$ [47]. Also our calculations may be of interest in the brane-world cosmological scenarios. The brane-world corresponds to a manifold with boundaries and all fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy, and as a result to the vacuum forces acting on the branes. In dependence of the type of a field and boundary conditions imposed, these forces can either stabilize or destabilize the brane-world. In addition, the Casimir energy gives a contribution to both the brane and bulk cosmological constant and, hence, has to be taken into account in the self-consistent formulation of the brane-world dynamics. In general, we believe the idea of Casimir effect in signature-changing space-time is novel and interesting. Therefore, the study of Casimir effect in the present model may provide important results relevant to the study of cosmological brane-world scenarios.

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